Ready to Preorder: Get Your BCCSP Axiomatization for Free! *

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Abstract. This paper contributes to the study of the equational theory of the semantics in van Glabbeek's linear time - branching time spectrum over the language BCCSP, a basic process algebra for the description of finite synchronization trees. It offers an algorithm for producing a complete (respectively, ground-complete) equational axiomatization of any behavioral congruence lying between ready simulation equivalence and partial traces equivalence from a complete (respectively, ground-complete) inequational axiomatization of its underlying precongruence—that is, of the precongruence whose kernel is the equivalence. The algorithm preserves finiteness of the axiomatization when the set of actions is finite.

1 Introduction

The lack of consensus on what constitutes an appropriate notion of observable behaviour for reactive systems has led to a large number of proposals for behavioural equivalences and preorders for concurrent processes. In his by now classic paper [13], van Glabbeek presented the linear time - branching time spectrum of behavioural preorders and equivalences for finitely branching, concrete, sequential processes. The semantics in this spectrum are based on simulation notions and on decorated traces.

Van Glabbeek [13] studied the semantics in his spectrum in the setting of the process algebra BCCSP, which contains only the basic process algebraic operators from CCS [18] and CSP [17], but is sufficiently powerful to express all finite synchronization trees. In the aforementioned reference, van Glabbeek gave, amongst a wealth of other results, (in)equational axiomatizations for the preorders and equivalences in the spectrum, such that two closed BCCSP terms can be equated by the axioms if, and only if, they are related by the preorder or equivalence in question. Groote [14] obtained ω-completeness results for most of

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the axiomatizations, in case the alphabet of actions is infinite. (An axiomatization E is *ω-complete* when an equation can be derived from E if, and only if, all of its closed instantiations can be derived from E.) The papers [2, 6, 8–10] offer positive and negative results on the existence of finite (in)equational axiomatizations for several behavioural equivalences and preorders in the spectrum over the language BCCSP, both in the setting of finite and infinite sets of actions.

The work we present in this paper stems from the observation that all of the extant axiomatization results presented in the aforementioned studies are based on separate, and often rather similar, developments for preorders and equivalences. For the semantics in the spectrum lying between 2-nested simulation semantics and partial traces semantics, the equivalences are the *kernels* of the preorders—meaning that two processes are considered equivalent if, and only if, each is a refinement of the other with respect to the preorder—, which are therefore more basic than the equivalences. Since the equivalences are defined in terms of the preorders in a canonical fashion, it would be very satisfying, in order to achieve a higher degree of generality and to highlight the commonalities in the technical developments pertaining to axiomatization results for the semantics in the spectrum, to develop a general strategy for obtaining complete axiomatizations of the equivalences in the spectrum from complete axiomatizations of the preorders. This is the aim of this paper.

Our contribution We offer an algorithm for producing an ω-complete (respectively, ground-complete) equational axiomatization of any behavioral congruence lying between ready simulation equivalence and partial traces equivalence from an ω-complete (respectively, ground-complete) inequational axiomatization of its underlying precongruence—that is, of the precongruence whose kernel is the equivalence. The algorithm we give in this paper preserves finiteness of the axiomatization when the set of actions is finite. It follows that each equivalence in the spectrum whose discriminating power lies in between that of ready simulation and partial traces equivalence is finitely axiomatizable over the language BCCSP if so is its defining preorder.

Our algorithm may be seen as isolating and axiomatizing the ingredients that all of the extant proofs of completeness results for the class of behavioural equivalences we study have in common. It also eliminates the need to reprove, essentially from scratch, completeness results for a large fragment of behavioural equivalences in the spectrum once a completeness result has been obtained for their underlying preorders. The axiomatizations that are automatically generated by our algorithm are very similar, when not identical, to those presented in the literature. (See, for instance, the two specific examples of applications of our algorithm that are provided in Section 6.)

Our algorithm takes as input a sound and ω-complete (respectively, ground-complete) inequational axiomatization E for BCCSP modulo a preorder in the linear time - branching time spectrum that includes the ready simulation preorder. Without loss of generality, we assume that the four classic equations from [16] that completely axiomatize bisimulation equivalence [18] are contained in E, and that so do the defining inequational axioms for ready simulation for each action a:

$$ax \preccurlyeq ax + ay \ .$$

The axiomatization $\mathcal{A}(E)$ generated by our algorithm from E contains the axioms for bisimulation equivalence together with the following equations, for each inequational axiom $t \preccurlyeq u$ in E:

- $t + u \approx u$; and
- $b(t + x) + b(u + x) \approx b(u + x)$ (for each action b, and some variable x that does not occur in $t + u$).

The main technical result in the paper is a theorem to the effect that the axiomatization $\mathcal{A}(E)$ is ω-complete (respectively, ground-complete) for the equivalence if E is ω-complete (respectively, ground-complete) for the preorder (Theorem 1). The proof of this statement is non-trivial, and relies on a careful analysis of the so-called *cover equations* [10] for the semantics in the linear time - branching time spectrum we consider in this study. Cover equations give us an explicit description of the equational theory for a particular semantics in terms of equations having a rather simple, and canonical, form.

Roadmap of the paper The paper is organized as follows. Section 2 reviews the syntax and the operational semantics for the language BCCSP, introduces the linear time time - branching time spectrum, and discusses the very basic notions of (in)equational logic used in this study. We present our algorithm in Section 3, where we also state the main theorem in the paper (Theorem 1) to the effect that the algorithm is guaranteed to produce an ω-complete (respectively, ground-complete) equational axiomatization of any behavioral congruence lying between ready simulation equivalence and partial traces equivalence from an ω-complete (respectively, ground-complete) inequational axiomatization of its underlying precongruence. The bulk of the rest of the paper (Sections 4–5) is devoted to a proof of our main result. Section 6 presents applications of our algorithm in the setting of simulation and failures semantics. We end the paper with some concluding remarks, and a detailed comparison with related work (Section 7).

2 Preliminaries

Syntax of BCCSP BCCSP(A) is a basic process algebra for expressing finite process behaviour. Its syntax consists of closed (process) terms p, q that are constructed from a constant $\mathbf{0}$, a binary operator $\_+\_$ called *alternative composition*, and unary *prefix* operators $a\_$, where a ranges over some nonempty set A of *actions* (with typical elements a, b, c, d). (We write $|A|$ for the cardinality of the set A.) Open terms p, q, r, s, t, u can moreover contain occurrences of variables from a countably infinite set V (with typical elements w, x, y, z).

A (closed) substitution maps variables in V to (closed) terms. For every term t and (closed) substitution σ, the (closed) term $\sigma(t)$ is obtained by replacing every occurrence of a variable x in t by $\sigma(x)$. We often write t^σ in lieu of $\sigma(t)$.

A *context* $C[]$ is a BCCSP(A) term with exactly one occurrence of a hole $[]$ in it. For every context $C[]$ and term p, we write $C[p]$ for the term that results by placing p in the hole in $C[]$.

Transition rules Intuitively, closed BCCSP(A) terms represent finite process behaviours, where $\mathbf{0}$ does not exhibit any behaviour, $p+q$ is the nondeterministic choice between the behaviours of p and q, and ap executes action a to transform into p. This intuition is captured, in the style of Plotkin, by the transition rules below, which give rise to A-labelled transitions between closed terms.

$$\frac{}{ax \xrightarrow{a} x} \qquad \frac{x \xrightarrow{a} x'}{x+y \xrightarrow{a} x'} \qquad \frac{y \xrightarrow{a} y'}{x+y \xrightarrow{a} y'}$$

The operational semantics is extended to open terms by assuming that variables do not exhibit any behaviour.

Linear time - branching time spectrum Van Glabbeek [13] presented the linear time - branching time spectrum of behavioural preorders and equivalences; see Figure 1. The semantics in this spectrum are based on simulation notions and on decorated traces. In what follows, we use $\precsim$ to denote a preorder in this spectrum, and $\simeq$ to denote the corresponding equivalence (i.e., $\precsim \cap \precsim^{-1}$). The equivalence induced by a preorder is also known as its *kernel*. When we want to refer to a specific preorder in the spectrum, we shall subscribe the symbol $\precsim$ with the initials of the intended semantics. For instance, we shall use $\precsim_{\mathrm{RS}}$ to denote the ready simulation preorder, $\precsim_{\mathrm{S}}$ for the simulation preorder, $\precsim_{\mathrm{F}}$ for the failures preorder, $\precsim_{\mathrm{CT}}$ for the completed traces preorder, and $\precsim_{\mathrm{PT}}$ for the partial traces preorder. A similar notational convention applies to the kernels of the preorders.

Each preorder in the linear time - branching time spectrum is a *precongruence* over the algebra of closed BCCSP(A) terms. That is, $p_1 \precsim q_1$ and $p_2 \precsim q_2$ imply $ap_1 \precsim aq_1$, for each $a \in A$, and $p_1 + p_2 \precsim q_1 + q_2$. Likewise, the equivalences in the spectrum constitute a *congruence* over closed BCCSP(A) terms.

Given a preorder $\precsim$ over closed terms, for open terms t and u, we define $t \precsim u$ if $\rho(t) \precsim \rho(u)$ for each closed substitution ρ; the corresponding equivalence $\simeq$ is lifted to open terms likewise.

Equations and inequations An *(in)equational axiomatization* (often abbreviated to *axiomatization*) E is a collection of either inequations $t \preccurlyeq u$ or equations $t \approx u$, where t and u are BCCSP(A) terms. We write $E \vdash t \preccurlyeq u$ or $E \vdash t \approx u$ if this (in)equation can be derived from the (in)equations in E using the standard rules of (in)equational logic, where the rule for symmetry can be applied for equational derivations but not for inequational ones. An axiomatization E is *sound* modulo $\precsim$ (or $\simeq$) if, for all open terms t, u, from $E \vdash t \preccurlyeq u$ (or $E \vdash t \approx u$) it follows that $t \precsim u$ (or $t \simeq u$). An axiomatization E is *ground-complete* modulo $\precsim$ (or $\simeq$) if $p \precsim q$ (or $p \simeq q$) implies $E \vdash p \preccurlyeq q$ (or $E \vdash p \approx q$), for all closed terms p and q. We say that E is *ω-complete* if for all open terms t, u with $E \vdash \rho(t) \preccurlyeq \rho(u)$ (or $E \vdash \rho(t) \approx \rho(u)$) for all closed substitutions ρ, we have $E \vdash t \preccurlyeq u$ (or $E \vdash t \approx u$).

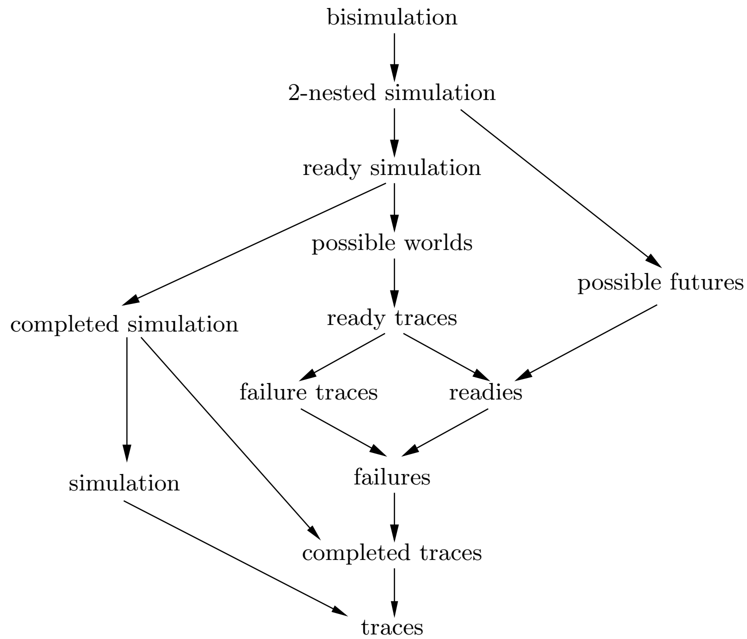

Fig. 1. The linear time - branching time spectrum

The core axioms A1–4 for BCCSP(A) given below are ω-complete [19], and sound and ground-complete [16, 18] modulo bisimulation equivalence, which is the finest semantics in the linear time - branching time spectrum.

$$\begin{array}{ll} \text{A1} & x + y \approx y + x \\ \text{A2} & (x + y) + z \approx x + (y + z) \\ \text{A3} & x + x \approx x \\ \text{A4} & x + \mathbf{0} \approx x \end{array}$$

In the remainder of this paper, process terms are considered modulo A1–4. A term x or at is a *summand* of each term $x + u$ or $at + u$, respectively. We use *summation* $\sum_{i=1}^{n} t_i$ (with $n \geq 0$) to denote $t_1 + \cdots + t_n$, where the empty sum denotes $\mathbf{0}$. As binding convention, alternative composition and summation bind more weakly than prefixing. Modulo the equations A1–4 each BCCSP(A) term t can be written in the form $\sum_{i=1}^{n} t_i$, where each t_i is either a variable or is of the form at' for some action a and term t'.

In his paper [13], van Glabbeek offered, amongst a host of other results, (in)equational axiomatizations for the preorders and equivalences in the spectrum. The proofs of the completeness results in that reference mostly employ the method of graph transformations. Groote [14] obtained ω-completeness results for most of the axiomatizations, in case the alphabet of actions is infinite.

In the remainder of this paper, in case of an infinite alphabet, occurrences of action names in axioms should be interpreted as action variables.

3 Producing an Axiomatization

Consider a preorder $\precsim$ in the linear time - branching time spectrum that includes the ready simulation preorder. Let E be a sound and ground-complete inequational axiomatization for BCCSP(A) modulo $\precsim$. We give an algorithm to produce an axiomatization $\mathcal{A}(E)$ that is sound and ground-complete for BCCSP(A) modulo $\simeq$, namely the kernel of the preorder $\precsim$. Moreover, if E is ω-complete, then so is $\mathcal{A}(E)$.

Without loss of generality, we assume that the axioms A1–4 are present in E, together with the defining inequational axioms for ready simulation equivalence for each $a \in A$:

$$ax \preccurlyeq ax + ay \ \ .$$

The axiomatization $\mathcal{A}(E)$ is constructed as follows. The axioms A1–4 are by default included in $\mathcal{A}(E)$. Furthermore, for each inequational axiom $t \preccurlyeq u$ in E, we add to $\mathcal{A}(E)$:

A. $t + u \approx u$; and
B. $b(t + x) + b(u + x) \approx b(u + x)$ (for all $b \in A$, and some x that does not occur in $t + u$).

Note that $\mathcal{A}(E)$ is finite whenever A and E are finite. Moreover, using an action variable in step B in lieu of a concrete action $b \in A$, the axiomatization $\mathcal{A}(E)$ contains only finitely many axiom schemas when E does, even in the presence of an infinite collection of actions.

Remark 1. Since $ax \preccurlyeq ax + ay$ is assumed to be present in E for each $a \in A$, by step B of the algorithm, the defining axioms for ready simulation from [6], namely

$$b(ax + z) + b(ax + ay + z) \approx b(ax + ay + z) \ \ ,$$

are present in $\mathcal{A}(E)$, for all $a, b \in A$.

We are now ready to present the main result of the paper to the effect that the algorithm defined above delivers axiomatizations for the kernels of the preorders that are sound, and ground- or ω-complete.

Theorem 1. *Let $\precsim$ be a preorder in the linear time - branching time spectrum with $\precsim_{\mathrm{RS}} \subseteq \precsim$. Let E be a sound and ground-complete inequational axiomatization for* BCCSP(A) *modulo $\precsim$. Then the equational axiomatization $\mathcal{A}(E)$ is sound and ground-complete for* BCCSP(A) *modulo $\simeq$. Moreover, if E is ω-complete, then so is $\mathcal{A}(E)$.*

Since the algorithm presented above preserves finiteness of the axiomatization when the set of actions A is finite, it follows that each equivalence in the spectrum whose discriminating power lies in between that of ready simulation and partial traces equivalence is finitely axiomatizable over the language BCCSP(A) if so is its defining preorder.

The remainder of the paper will be essentially devoted to a proof of the above theorem. Our proof of Theorem 1 relies on the isolation of a collection of equations, the so-called *cover equations*, that have a simple form and completely characterize the equational theory of BCCSP(A) modulo any of the behavioural equivalences whose discriminating power lies in between that of ready simulation and partial traces equivalence. Restricting ourselves to cover equations will help us overcome the technical complications in the proof-theoretic argument we shall use in Section 5 to complete the proof of Theorem 1.

In light of the key role cover equations play in the proof of Theorem 1, we now proceed to introduce them and to analyze the properties that make them a crucial ingredient in our proof of that result.

4 Cover Equations

For bisimulation semantics, and thus for all process semantics in the linear time - branching time spectrum, axiom A3 is sound. So if an equation $t \approx u$ is sound, then $u + t \approx t$ and $t + u \approx u$ are sound too; and from the last two equations one can derive $t \approx u$. Furthermore, for all process semantics in the linear time - branching time spectrum, if $t_1+t_2+u \approx u$ is sound, then $t_1+u \approx u$ and $t_2+u \approx u$ are sound; and from the last two equations one can derive $t_1+t_2+u \approx u$. Hence, from the point of view of provability, it suffices only to consider sound equations of the form $at + u \approx u$ and $x + u \approx u$. We call these the *cover equations*. We present three lemmas that limit the form that cover equations can have for the semantics in the spectrum we study in this paper. (In the statements of the lemmas below, t and u range over the collection of open BCCSP(A) terms.)

Lemma 1. *If $t + x \precsim u$, and either $\precsim \subseteq \precsim_{\mathrm{CT}}$, or $\precsim \subseteq \precsim_{\mathrm{PT}}$ and $|A| > 1$, then x is a summand of u.*

If $|A| = 1$, then the partial traces preorder and the simulation preorder coincide—see, e.g., [3]. For this special case, Lemma 1 fails. Namely, let $A = \{a\}$. Then $x \precsim ax$ is sound for the partial traces (and simulation) preorder.

Lemma 2. *Let $\simeq$ be an equivalence in the linear time - branching time spectrum. If $at + u + bv \simeq u + bv$ with $a \neq b$, then $at + u \simeq u$.*

This lemma is trivial to check for each of the equivalences in the linear time - branching time spectrum. The key idea is that since $a \neq b$, the non-empty (decorated) traces of at and bu are disjoint, and bu cannot (ready/completed) simulate at.

The following lemma states a kind of cancellation result for the preorders in the spectrum.

Lemma 3. *Let $\precsim$ be a preorder in the linear time - branching time spectrum. If $t + x \precsim u + x$, and x is not a summand of $t + u$, then $t \precsim u$.*

The condition in Lemma 3 that x is not a summand of $t+u$ is essential. For instance, $x+x \precsim_{\mathrm{PT}} \mathbf{0}+x$, but $x \not\precsim_{\mathrm{PT}} \mathbf{0}$. And $\mathbf{0}+x \precsim_{\mathrm{CT}} x+x$, but $\mathbf{0} \not\precsim_{\mathrm{CT}} x$.

Lemma 3 needs to be proved separately for each preorder in the linear time - branching time spectrum. Despite the naturalness of its statement, which appears obvious, these proofs are not trivial, and quite technical. Fokkink and Nain [10] proved such a lemma for failures semantics, with the aim to obtain an ω-completeness result for this semantics, and their proof is rather delicate. The details of the proof of Lemma 3 can be found in the full version of this paper [4].

From the three lemmas above, one can conclude that in order to prove ω-completeness (or ground-completeness) of an equational axiomatization, it suffices to derive all sound equations (or all sound closed equations) of the form

$$at + \sum_{i=1}^{n} au_i \approx \sum_{i=1}^{n} au_i \qquad (n \geq 1)$$

and, only for the case of partial traces semantics with $|A| = 1$, all sound equations of the form

$$x + u \approx u \ .$$

In our proof of Theorem 1, we shall therefore focus on showing that the equational axiomatization $\mathcal{A}(E)$ generated by our algorithm is powerful enough to prove all of the sound equations of the above two forms.

5 Proof of Theorem 1

Proof. Let $\precsim$ be a preorder in the linear time - branching time spectrum, with $\precsim_{\mathrm{RS}} \subseteq \precsim$. Let E be a sound and ground-complete inequational axiomatization for BCCSP(A) modulo $\precsim$.

It is not hard, albeit tedious, to see that the equational axiomatization $\mathcal{A}(E)$ is sound for BCCSP(A) modulo $\simeq$. We prove that ω-completeness of E implies ω-completeness of $\mathcal{A}(E)$. The proof that $\mathcal{A}(E)$ is ground-complete is identical, but assumes that all terms that occur in the proof below are closed. (It is well known that if an axiomatization proves a closed (in)equation, then there is a closed proof for that (in)equation.)

We note that, for each of the preorders in the linear time - branching time spectrum, $ar + as + t \precsim u$ if, and only if, both $ar + t \precsim u$ and $as + t \precsim u$. This, together with the presence of the axiom A3, implies that the inequational axiomatization E that we start with can be pre-processed so that there are no multiple a-summands on the left-hand sides of the inequational axioms in E.

Moreover, in view of Lemmas 1 and 3, if $\precsim \subseteq \precsim_{\mathrm{CT}}$ or $|A| > 1$, then variable summands on the left-hand sides of inequational axioms can be omitted. Concluding, in this case we can assume that the inequational axiomatization E that we start with only contains inequational axioms of the form $ap \preccurlyeq \sum_{i=1}^{n} aq_i$ (with $n \geq 1$) or $\mathbf{0} \preccurlyeq q$.

For the case of partial traces semantics with $|A| = 1$, Lemma 1 does not apply. Note, however, that $r + s \precsim_{\mathrm{PT}} u$ if, and only if, both $r \precsim_{\mathrm{PT}} u$ and $s \precsim_{\mathrm{PT}} u$.

Hence, for this special case it suffices to allow also for inequational axioms of the form $x \preccurlyeq q$.

We start with showing that all cover equations of the form $at + u \approx u$ can be derived from $\mathcal{A}(E)$. (Cover equations of the form $x + u \approx u$ will be considered later.) In view of Lemmas 2 and 3, it suffices to only consider those equations where u is of the form $\sum_{i=1}^{n} au_i$ with $n \geq 1$. Let

$$at + \sum_{i=1}^{n} au_i \simeq \sum_{i=1}^{n} au_i \ .$$

We show that the corresponding cover equation can be derived from $\mathcal{A}(E)$. It is not hard to see that, for the semantics in the linear time - branching time spectrum, the above equivalence implies

$$at \precsim \sum_{i=1}^{n} au_i \ .$$

So by ω-completeness of E,

$$E \vdash at \preccurlyeq \sum_{i=1}^{n} au_i \ .$$

We prove, using induction on the length of such a derivation, not counting applications of axioms A1–4, that

$$\mathcal{A}(E) \vdash at + \sum_{i=1}^{n} au_i \approx \sum_{i=1}^{n} au_i \ .$$

Base case: $t = u_i$ for some i. Trivial using A1–3.

Inductive case: We distinguish two cases, which deal with instantiations of inequational axioms in context.

Case 1: The first step of the derivation is

$$E \vdash aC[p^\sigma] \preccurlyeq aC[q^\sigma] \ .$$

That is, $t = C[p^\sigma]$ for some context $C[]$, substitution σ, and inequational axiom $p \preccurlyeq q$. Then clearly $aC[p^\sigma]$ is of the form $D[b(p^\sigma + r)]$ and $aC[q^\sigma]$ is of the form $D[b(q^\sigma + r)]$ for some context $D[]$, action b, and term r.

Since $E \vdash aC[q^\sigma] \preccurlyeq \sum_{i=1}^{n} au_i$ by a shorter derivation, by induction,

$$\mathcal{A}(E) \vdash aC[q^\sigma] + \sum_{i=1}^{n} au_i \approx \sum_{i=1}^{n} au_i \ .$$

Furthermore,

$$\mathcal{A}(E) \vdash aC[p^\sigma] + aC[q^\sigma] \approx aC[q^\sigma] \ .$$

This equation can indeed be derived from the axiom $b(p+x)+b(q+x) \approx b(q+x)$, which is present in $\mathcal{A}(E)$ for each $b \in A$ according to step B in the algorithm, together with the defining axiom for ready simulation, $b(cx+z)+b(cx+cy+z) \approx b(cx+cy+z)$, which by assumption is present in $\mathcal{A}(E)$ for all $b,c \in A$ (see Remark 1). The derivation of the above equation is by induction on the depth of the occurrence of the context symbol $[]$ within $C[]$.

- Let $[]$ occur at depth zero in $C[]$, i.e., $C[] = [] + r$ for some term r. Let the substitution ρ coincide with σ on variables in p and q, and let $\rho(x) = r$. (Recall that an assumption in step B of the algorithm was that x does not occur in $p+q$.) The derivation simply consists of applying the substitution ρ to the axiom $a(p+x)+a(q+x) \approx a(q+x)$.
- Let $C[] = dC'[] + s$. By induction on the depth of the occurrence of $[]$, $\mathcal{A}(E) \vdash dC'[p^\sigma] + dC'[q^\sigma] \approx dC'[q^\sigma]$. So

$$\begin{array}{rcl}\mathcal{A}(E) & \vdash & aC[p^\sigma] + aC[q^\sigma] \;=\; a(dC'[p^\sigma]+s) + a(dC'[q^\sigma]+s) \\ & \approx & a(dC'[p^\sigma]+s) + a(dC'[p^\sigma]+dC'[q^\sigma]+s) \\ & \approx & a(dC'[p^\sigma]+dC'[q^\sigma]+s) \\ & \approx & a(dC'[q^\sigma]+s) \;=\; aC[q^\sigma] \ .\end{array}$$

Hence,

$$\begin{array}{rcl}\mathcal{A}(E) & \vdash & aC[p^\sigma] + \sum_{i=1}^{n} au_i \;\approx\; aC[p^\sigma] + aC[q^\sigma] + \sum_{i=1}^{n} au_i \\ & \approx & aC[q^\sigma] + \sum_{i=1}^{n} au_i \;\approx\; \sum_{i=1}^{n} au_i \ ,\end{array}$$

which was to be shown.

CASE 2: The first step of the derivation is

$$E \vdash ap^\sigma \preccurlyeq \sum_{j=1}^{m} aq_j^\sigma \qquad (m \geq 1) \ .$$

That is, $t = p^\sigma$ for some substitution σ and inequational axiom $ap \preccurlyeq \sum_{j=1}^{m} aq_j$.

By the soundness of E, clearly $aq_j^\sigma \precsim \sum_{i=1}^{n} au_i$ for $j = 1, \ldots, m$. So by ω-completeness, $E \vdash aq_j^\sigma \preccurlyeq \sum_{i=1}^{n} au_i$ for $j = 1, \ldots, m$. By one of our assumptions, the inequational axioms in E do not contain multiple occurrences of a-summands on their left-hand sides. This implies that each of these derivations is not longer than the derivation of $E \vdash \sum_{j=1}^{m} aq_j^\sigma \preccurlyeq \sum_{i=1}^{n} au_i$. So by induction,

$$\mathcal{A}(E) \vdash aq_j^\sigma + \sum_{i=1}^{n} au_i \approx \sum_{i=1}^{n} au_i$$

for $j = 1, \ldots, m$. Furthermore, according to step A of the algorithm, the axiom $p + \sum_{j=1}^{m} aq_j \approx \sum_{j=1}^{m} aq_j$ is present in $\mathcal{A}(E)$. Hence,

$$\begin{aligned}\mathcal{A}(E) \;\; \vdash \;\; & ap^\sigma + \sum_{i=1}^{n} au_i \;\approx\; ap^\sigma + \sum_{j=1}^{m} aq_j^\sigma + \sum_{i=1}^{n} au_i \\ \approx \;\; & \sum_{j=1}^{m} aq_j^\sigma + \sum_{i=1}^{n} au_i \;\approx\; \sum_{i=1}^{n} au_i \;\; .\end{aligned}$$

This completes the proof for the case of cover equations of the form $at + \sum_{i=1}^{n} au_i \simeq \sum_{i=1}^{n} au_i$.

It remains to prove that cover equations of the form $x + u \approx u$ can be derived from $\mathcal{A}(E)$. If $\precsim \subseteq \precsim_{\rm CT}$ or $|A| > 1$, then in view of Lemma 1, such cover equations can be derived using A3. So we are left to consider the special case that $\precsim = \precsim_{\rm PT}$ and $|A| = 1$. Let

$$x + u \simeq_{\rm PT} u \;\; .$$

Clearly, this implies

$$x \precsim_{\rm PT} u \;\; .$$

So, by ω-completeness of E,

$$E \vdash x \preccurlyeq u \;\; .$$

We prove, using induction on the length of such a derivation, not counting applications of A1–4, that

$$\mathcal{A}(E) \vdash x + u \approx u \;\; .$$

Base case: x is a summand of u. Trivial.

Inductive case: The first step of the derivation is

$$E \vdash y^\sigma \preccurlyeq q^\sigma \;\; .$$

That is, $\sigma(y) = x$ for some substitution σ and inequational axiom $y \preccurlyeq q$ in E.

By the soundness of E, clearly $r \precsim_{\rm PT} u$ for each summand r of q^σ. So by ω-completeness, $E \vdash r \preccurlyeq u$. By assumption, the inequational axioms in E are all of the form $as \preccurlyeq \sum_{i=1}^{n} as_i$ (with $n \geq 1$) or $\mathbf{0} \preccurlyeq s$ or $z \preccurlyeq s$, for some variable z. This implies that each of these derivations is not longer than the derivation of $E \vdash q^\sigma \preccurlyeq u$. So by induction and A3,

$$\mathcal{A}(E) \vdash q^\sigma + u \approx u \;\; .$$

Furthermore, according to step A of the algorithm, the axiom $y + q \approx q$ is present in $\mathcal{A}(E)$. Hence,

$$\mathcal{A}(E) \;\vdash\; y^\sigma + u \;\approx\; y^\sigma + q^\sigma + u \;\approx\; q^\sigma + u \;\approx\; u \;\; .$$

The proof of the theorem is now complete. □

6 Examples

We show how our algorithm produces equational axiomatizations for two equivalences in the linear time - branching time spectrum—namely simulation and failures—from the inequational axiomatizations for the corresponding preorders. For the simulation preorder, we leave out the pre-supposed inequational axiom $ax \preccurlyeq ax + ay$, since it can be derived from the defining inequational axioms for that preorder.

6.1 Simulation

Let $|A| > 1$. Then A1–4 plus one inequational axiom

$$\mathbf{0} \preccurlyeq x$$

is a sound and ground-complete axiomatization for BCCSP(A) modulo the simulation preorder [13].

Step A of the algorithm produces the already present axiom A4:

$$\mathbf{0} + x \approx x \ .$$

Step B of the algorithm produces the defining axioms for simulation equivalence for each $b \in B$:

$$b(\mathbf{0} + y) + b(x + y) \approx b(x + y) \ .$$

6.2 Failures

Let $|A| \geq 1$. The axiomatization consisting of A1–4 plus one inequational axiom

$$a(x + y) \preccurlyeq ax + a(y + z)$$

for each $a \in A$ is sound and ground-complete for BCCSP(A) modulo the failures preorder [13].

Step A of the algorithm produces, for all $a \in A$:

$$a(x + y) + ax + a(y + z) \approx ax + a(y + z) \ .$$

This axiom is one of the two defining axioms for failures equivalence. (The second defining axiom for failures equivalence is the ready simulation axiom, which is assumed to be present from the start.)

Step B of the algorithm produces, for all $a, b \in A$:

$$b(a(x + y) + w) + b(ax + a(y + z) + w) \approx b(ax + a(y + z) + w) \ .$$

This axiom is redundant; it can be derived from the other axioms as follows. (The subterm to which an axiom is applied is underlined.)

$$
\begin{array}{ll}
 & b(\underline{ax + a(y+z)} + w) \\
\approx & \underline{b(a(x+y) + ax + a(y+z) + w)} \\
\approx & \underline{b(a(x+y) + a(y+z) + w)} + b(a(x+y) + ax + a(y+z) + w) \\
\approx & b(a(x+y) + w) + \underline{b(a(x+y) + a(y+z) + w) + b(a(x+y) + ax + a(y+z) + w)} \\
\approx & b(a(x+y) + w) + b(\underline{a(x+y) + ax + a(y+z)} + w) \\
\approx & b(a(x+y) + w) + b(ax + a(y+z) + w)
\end{array}
$$

7 Conclusions and Comparison with Related Work

In this paper, we have offered an algorithm for generating a ground-complete (respectively, ω-complete) axiomatization for behavioural equivalences in the linear time - branching time spectrum starting from a ground-complete (respectively, ω-complete) axiomatization for their underlying preorders—that is, of the preorders that have the equivalences as their kernels. Our algorithm applies to all of the process semantics in the spectrum whose discriminating power lies in between that of ready simulation semantics and of partial traces semantics. Moreover, in the presence of a finite set of actions, our procedure preserves finiteness of axiomatizations, and thus can be used to obtain automatically finite basis results for behavioural equivalences in the spectrum from similar results for their underlying preorders. In fact, our results apply to *any* behavioural precongruence whose discriminating power lies in between that of the ready simulation preorder and of the partial traces preorder, provided that Lemmas 1–3 hold for the precongruence in question.

Our algorithm may thus be considered as isolating and axiomatizing the ingredients that all of the extant proofs of completeness results for the class of behavioural equivalences we study have in common. (See, for example, the references [5, 6, 8–10, 13, 14] for a sample of such results.) It also eliminates the need to reprove, essentially from scratch, completeness results for a large fragment of behavioural equivalences in the spectrum once a completeness result has been obtained for their underlying preorders. As witnessed by the examples we provided in Section 6, the axiomatizations that are automatically generated by our algorithm are very similar, when not identical, to those presented in the literature. In this respect, this study may be seen as a companion to [1]. That paper offered an algorithm that generates a finite, ground-complete axiomatization for bisimulation equivalence from an operational specification of a language in GSOS format [7]. That procedure relies on the axiomatization of bisimulation equivalence over the language BCCSP. Here we have focused on the algorithmic generation of complete axiomatizations for other equivalences in the spectrum over the language BCCSP.

The spirit of our study is also very similar to the one in [12]. In that reference, independent of our work and building on their previous paper [11], de Frutos-Escrig and Gregorio-Rodríguez show, amongst other things, how to generate an inequational axiomatization for preorders in the spectrum from equational axiomatizations for the corresponding equivalence. They generate this inequational axiomatization by simply adding the defining inequational axioms for the ready simulation preorder to the axiomatization for the equivalence—see Theorem 5.1 in [12]. That result applies to behavioural equivalences in the linear time - branching time spectrum that (1) include ready simulation equivalence, and (2) whose underlying preorders only equate processes having the same set of initial actions. That second condition is not met by completed simulation, simulation, completed traces and partial traces semantics. Furthermore, the result from [12] only applies to ground-complete axiomatizations.

There are some interesting general connections between the technical developments in this paper and those in [12]. For instance, Lemma 3.11 in [12] gives a soundness proof for the equations generated by step A in our algorithm for the preorders in the spectrum that satisfy condition 2 above. However, the equations generated by step A are sound also for completed simulation, simulation, completed traces and partial traces semantics. So Lemma 3.11 in [12] is not as general as it could be.

It would also be interesting to investigate the possible relation between the cover equations approach, used in this paper to reduce the class of equations to be considered in the proof of completeness, and the condition of action factorization mentioned in the statement of Theorem 2.6 of [12]. (Action factorization means that if $p \precsim q$, then, for each action a, the sum of the a-summands of p is also dominated by the sum of the a-summands of q with respect to $\precsim$.)

In summary, our work differs from [12] in the following fundamental ways.

- We show how to produce an equational axiomatization for an equivalence from an inequational axiomatization of its underlying preorder. Since the equivalences in the linear time - branching time spectrum that include ready simulation equivalence are the kernels of their underlying preorders, to our mind, the preorders are a more basic notion to build on in this setting.
- Unlike Theorem 5.1 of [12], our main result applies to all of the semantics in the spectrum whose discriminating power lies in between that of ready simulation semantics and partial traces semantics.
- Unlike Theorem 5.1 of [12], our results apply to ω-complete as well as to ground-complete axiomatizations.

It would be interesting to extend our algorithm so that it applies also to nested simulation semantics [15] and to possible futures semantics [20]. However, as shown in [2], unlike the semantics we have considered in this study, nested simulation and possible futures semantics afford no finite ground-complete axiomatization over BCCSP even in the presence of a single action. This indicates that such a generalization of our results will not be easy to achieve without recourse to conditional equations. We leave such generalizations of our results and proof techniques as a topic for future investigations.

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