A relation between partitions and the number of divisors

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A sum of positive natural numbers adding up to n is called a *partition* of n. For instance, $1+2+4$ is a partition of 7. As none of the summands $1, 2, 4$ are equal, this is called a partition *into unequal parts*. There are five partitions of 7 into unequal parts:

$$1+2+4, \quad 1+6, \quad 2+5, \quad 3+4, \quad 7.$$

Since the partitions $1+2+4$ and 7 contain an odd number of summands, they are called *odd* partitions, whereas the other three partitions are called *even*. Add the smallest numbers of the odd partitions, $1+7=8$, and do the same for the smallest numbers of the even partitions, $1+2+3=6$. The difference between these two sums, $8-6=2$, is exactly the number of divisors of the prime 7.

In the sequel, $p(n)$ denotes the sum of the smallest numbers of odd partitions of n minus the smallest numbers of even partitions of n, and $d(n)$ denotes the number of divisors of n. For small numbers n, it is easy to check that $p(n)$ equals $d(n)$. This is not a coincidence; we shall see that it is a general relation between the smallest numbers of partitions into unequal parts and the number of divisors.

Theorem. $p(n) = d(n)$ for all positive natural numbers n.

In order to prove this theorem, we introduce the sum of polynomial quotients

$$P_n(X) \;=\; \sum_{i=0}^{n-1} \frac{(1-X^{i+1})(1-X^{i+2})\ldots(1-X^n)}{1-X^{n-i}}$$

for positive natural numbers n. At each consecutive quotient, the degree of the denominator decreases by one, and the leftmost factor in the numerator drops out. Fix an $m = 1, ..., n$. We shall show that the coefficient α_m for X^m in $P_n(X)$ equals $d(m) - p(m)$.

First, we determine the contributions from the separate quotients of $P_n(X)$ to α_m. Fix an $i = 0, ..., n-1$, and replace the denominator $1/(1-X^{n-i})$ in the ith quotient of $P_n(X)$ by its power series (which converges for $|X| < 1$). Hence, the ith quotient of $P_n(X)$ takes the form

$$(1-X^{i+1})\ldots(1-X^n)(1+X^{n-i}+X^{2(n-i)}+\ldots).$$

Since $m \leq n$, the contributions from this product to α_m stem either from $(1-X^{i+1})\ldots(1-X^n)$ or from $(1+X^{n-i}+X^{2(n-i)}+\ldots)$. Now, we collect the contributions to α_m of these two types of terms.

1. Clearly, the series $(1 + X^{n-i} + X^{2(n-i)} + \ldots)$ contributes $+1$ to the coefficient α_m of X^m if and only if $n - i$ is a divisor of m.

   As i increases from 0 to $n - 1$, the number $n - i$ decreases from n to 1. In this range there are $d(m)$ numbers which divide m, so there are $d(m)$ series $(1+X^{n-i}+X^{2(n-i)}+\ldots)$ for $i = 0, ..., n-1$ which contribute $+1$ to α_m. These contributions together sum up to $d(m)$.

2. If we decompose the product $(1 - X^{i+1}) \ldots (1 - X^n)$, this results into terms $(-1)^l X^{k_1+\ldots+k_l}$ for all sequences of numbers $i+1 \leq k_1 < \ldots < k_l \leq n$. So this product contributes $+1$ to α_m for each even partition of m with terms greater than i, and it contributes -1 to α_m for each odd partition of m with terms greater than i.

   So for each even partition of m with smallest term k, the products $(1 - X^{i+1}) \ldots (1 - X^n)$ for $i = 0, ..., k - 1$ contribute $+1$ to α_m. These contributions together sum up to k.

   Likewise, for each odd partition of m with smallest term k, the products $(1 - X^{i+1}) \ldots (1-X^n)$ for $i = 0, ..., k-1$ contribute -1 to α_m. These contributions together sum up to $-k$.

   So in total, these contributions to α_m sum up to $-p(m)$.

Hence, we have found that α_m equals $d(m) - p(m)$ for $m = 1, ..., n$. So to prove our main theorem, it suffices to prove the following proposition.

Proposition. $P_n(X)$ equals n for all $n \geq 1$.

Proof. Note that $P_1(X) = 1$. To prove the proposition, we show that the difference between $P_{n+1}(X)$ and $P_n(X)$ is equal to 1.

Shifting the index i of the sum $P_n(X)$ by one, $P_{n+1}(X) - P_n(X)$ takes the form

$$\sum_{i=0}^{n} \frac{(1 - X^{i+1}) \ldots (1 - X^{n+1})}{1 - X^{n+1-i}} - \sum_{i=1}^{n} \frac{(1 - X^{i}) \ldots (1 - X^{n})}{1 - X^{n+1-i}}.$$

In the second sum, we can also start the index i at 0, because its quotient for $i = 0$ equals zero. Now, collecting quotients of equal denominator gives

$$\sum_{i=0}^{n} (X^i - X^{n+1}) \frac{(1 - X^{i+1}) \ldots (1 - X^{n})}{1 - X^{n+1-i}}.$$

The denominator $1 - X^{n+1-i}$ divides the factor $X^i - X^{n+1}$ in the numerator, so this sum equals

$$\sum_{i=0}^{n} X^i (1 - X^{i+1}) \ldots (1 - X^n).$$

Denote this polynomial by $Q_n(X)$. The result will follow if $Q_n(x) = 1$ for all n. Again, we use induction. $Q_1(x) = (1 - X) + X = 1$, and isolating the term with $i = n + 1$ in the expression for $Q_{n+1}(x)$ yields the relation $Q_{n+1}(x) = (1 - X^{n+1}) Q_n(x) + X^{n+1}$, which provides the inductive step. □

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References

[1] J. Peeters and O. Ciftcioglu. Statistics on exponential averaging of periodograms. Report ECN-RX-94-010, Euratomic Centre, Petten, The Netherlands, 1994. To appear in *IEEE Transactions on Signal Processing*.

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